e-content for students

B. Sc.(honours) Part 1paper 2

Subject:Mathematics

Topic:Rectangular, Cylindrical & Spherical

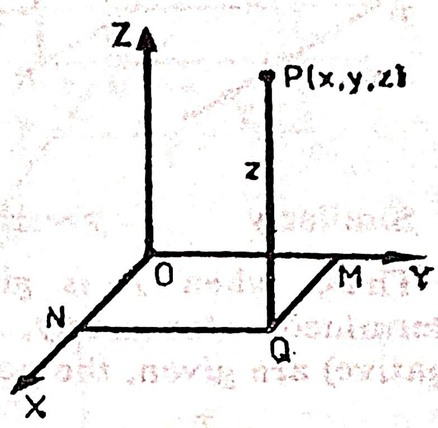
Co-ordinates

RRS college mokama

Rectangular Cartesian co-ordinates

Let OX, OY, OZ be three mutually perpendicular straight lines in space.

Clearly they are the lines of intersection of three pairs of planes $XOY, XOZ; XOY, YOZ; ZOX, ZOY$ respectively. Let P be any point in space. From P draw PQ perpendicular to the XOY plane. From Q draw QM and QN perpendiculars to OY and OX respectively. Then ON, NQ, QP are the distances of the point P from the planes YOZ, XOZ, XOY respectively.



These distances are called the x, y, z co-ordinates of the point P and we write it as $P(x, y, z)$. It is interesting to note that the co-ordinates of the foot, Q, of the perpendicular from $P(x, y, z)$ to the XOY plane in two-dimensions are (x, y).

The lines OX, OY, OZ are called the co-ordinate axes; O is called the origin of co-ordinates and the planes YOZ, ZOX, XOY are called the x, y, z co-ordinate planes respectively.

If OX, OY, OZ are not mutually perpendicular, then we call them oblique axes. Unless otherwise stated, we use only rectangular axes.

Co-ordinates of a point on any one of the axes.

(i) If the point is on the x-axis, then clearly the y- and z-coordinates of the point will be zero, that is, the co-ordinates of the point will be of the type $(x, 0, 0)$.

(ii) Similarly, the co-ordinates of any point on the y-axis are of the type $(0, y, 0)$.

(iii) Lastly, the co-ordinates of any point on the z-axis are of the type $(0, 0, z)$.

Cylindrical polar Co-ordinates

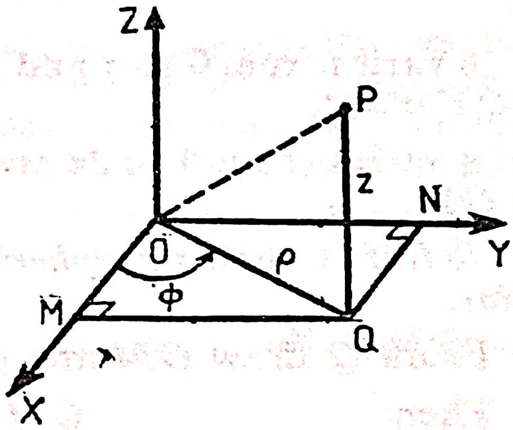
Let P be any point in space. From P draw PQ perpendicular to the XOY plane. Let P be (x, y, z) in rectangular Cartesian co-ordinates.

Then $PQ = z$.

Join O and Q.

Let $OQ = \rho$ and $\angle XOQ = \varphi$.

Then (ρ, φ, z) are called the cylindrical polar coordinates of P.



z is taken positive as long as P is above the XOY plane (on the side of OZ) and z is negative if P be below the XOY plane on the opposite side of OZ.

ρ is essentially positive and φ varies from 0 to 2π measured in the direction OX to OY.

Relations between cylindrical and rectangular co-ordinates of a point.

From Q draw QM and QN parallel to OY and OX respectively.

Then $OM = x$ and $ON = y$.

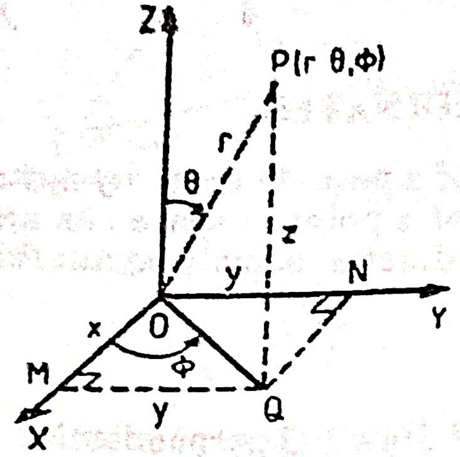
We have $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$ and $z = z$.

$\therefore \quad \rho = \sqrt{x^2 + y^2}$ and $\varphi = \tan^{-1} \dfrac{y}{x}$.

Hence $(x, y, z) \equiv (\rho\cos\varphi, \rho\sin\varphi, z)$

and $(\rho, \varphi, z) \equiv \left(\sqrt{x^2+y^2}, \tan^{-1}\dfrac{y}{x}, z\right)$

Spherical polar Co-ordinates



Let P be any point (x, y, z) in rectangular Cartesian co-ordinates.

Join O and P.

Let $OP = r$ and $\angle ZOP = \theta$.

From P draw PQ perpendicular to the plane of XOY.

Then $PQ = z$.

Join O and Q.

Let $\angle XOQ = \varphi$.

Then (r, θ, φ) are called the spherical polar coordinates of P.

Here we shall assume that r is always positive.

θ varies from 0 to π and is taken positive if measured from OZ to XOY plane.

φ varies from 0 to 2π and is taken positive if measured from OX to OY.

Relations between spherical polar and rectangular coordinates of a point.

From Q draw QM and QN parallel to OY and OX respectively.

Then $$OM = x \text{ and } ON = y.$$

$\therefore$ $OM = OQ\cos\varphi$ and $ON = OQ\sin\varphi$; also $\angle POQ = \dfrac{\pi}{2} - \theta$.

$$\therefore \quad OQ = OP\cos\left(\frac{\pi}{2} - \theta\right) = r\sin\theta,$$

and $$PQ = OP\sin\left(\frac{\pi}{2} - \theta\right) = r\cos\theta.$$

Hence $$x = OM = OQ\cos\varphi = r\sin\theta\cos\varphi,$$
$$y = ON = OQ\sin\varphi = r\sin\theta\sin\varphi, \; z = PQ = r\cos\theta.$$
$$\therefore \quad x^2 + y^2 = r^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi) = r^2\sin^2\theta$$

and $$x^2 + y^2 + z^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2.$$

Therefore $$r = \sqrt{x^2 + y^2 + z^2},$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z} \text{ and } \tan\varphi = \frac{y}{x}.$$

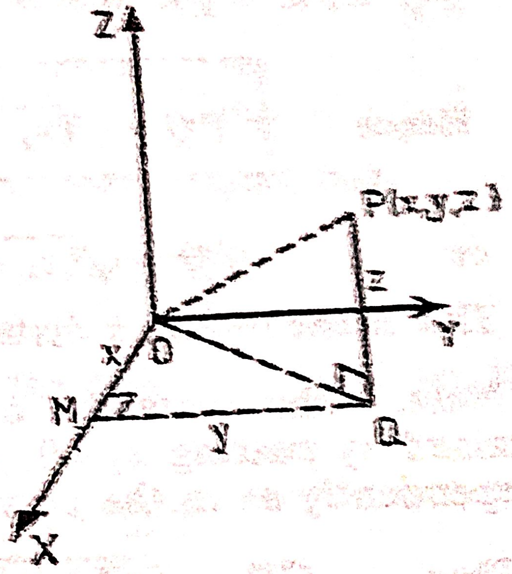
Hence $$(x, y, z) \equiv (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$$

and $$(r, \theta, \varphi) \equiv \left(\sqrt{x^2+y^2+z^2},\ \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\ \tan^{-1}\frac{y}{x}\right)$$

are the mutual relations between the rectangular and spherical coordinates of a point in space.

To find the distance of the point (x, y, z) from origin

Let P be the point (x, y, z). Join O and P. From P draw PQ perpendicular on XOY plane. Then $PQ = z$ and PQ is perpendicular to every line in the XOY plane; consequently $PQ \perp OQ$.



Through Q draw QM perpendicular to the axis of x.

Join O and Q.

Then $OM = x$ and $QM = y$.

From right angled triangle OMQ,

$OQ^2 = OM^2 + QM^2 = x^2 + y^2$.

Again from right angled triangle

OQP, $\quad OP = \sqrt{OQ^2 + PQ^2}$.

Hence $\quad \boxed{OP = \sqrt{x^2 + y^2 + z^2}}$.

This is the required distance.

Distance between two points

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the co-ordinates of the points P and Q respectively.

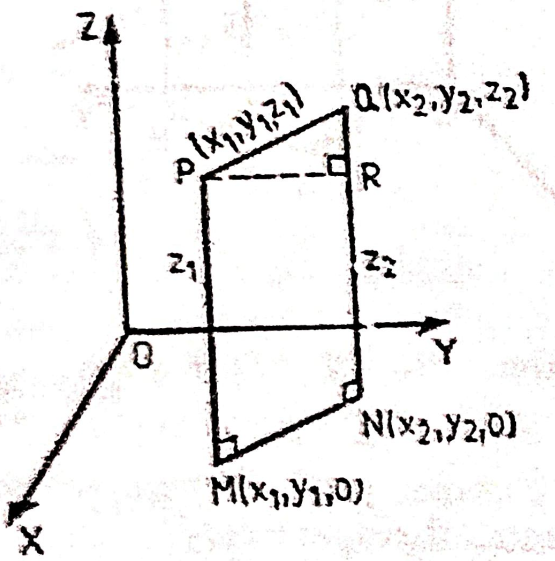
From P and Q draw PM and QN perpendiculars on the XOY plane. So $PM \parallel QN$ and

$$PM \perp MN, \; QN \perp MN.$$

Clearly P, M, N and Q are coplanar as $PM \parallel QN$.

Then $PM = z_1$ and $QN = z_2$.

The points M and N are respectively (x_1, y_1) and (x_2, y_2) (in the two-dimensional space i.e. the plane XOY).



$\therefore$ $MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, by Co-ordinate Geometry of two-dimensions.

Further, draw PR perpendicular to QN. Now in the plane $PMNR$, PR and MN are perpendicular to QN. So $PR \parallel MN$.

Thus $PMNR$ is a rectangle and so $RN = PM = z_1$.

Then $$QR = QN - RN = z_2 - PM = z_2 - z_1.$$

From right angled triangle PRQ,

$$PQ^2 = PR^2 + QR^2 = MN^2 + QR^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Hence $$\boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

or $$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This is the required distance.